# Linear "ship waves" generated in stationary flow of a Bose-Einstein condensate past an obstacle

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Using stationary solutions of the linearized two-dimensional Gross-Pitaevskii equation, we describe the "ship wave" pattern occurring in the supersonic flow of a Bose-Einstein condensate past an obstacle. It is shown that these "ship waves" are generated outside the Mach cone. The developed analytical theory is confirmed by numerical simulations of the flow past body problem in the frame of the full non-stationary Gross-Pitaevskii equation.

## INTRODUCTION

Experimental creation of Bose-Einstein condensate (BEC) has led to emergence of a new field of nonlinear wave dynamics owing to a remarkable richness of nonlinear wave patterns supported by this medium. First, vortices and bright and dark solitons were observed and their dynamics was studied theoretically in framework of the mean-field approach (see, e.g. [1] and references therein). Then, dispersive shocks generated by a large and sharp disturbance of BEC were found in experiment [2, 3] and explained theoretically [3, 4] in framework of the Whitham theory of modulations of nonlinear waves (see also numerical experiment in [5]). At last, the stationary waves generated by supersonic flow of BEC past obstacles have been recently observed [6]. They were studied in [7, 8, 9] where the main focus was on the nonlinear component representing a train of solitons (in the simplest case of a single soliton [8]) or, more precisely, having a form of a modulated nonlinear periodic wave. The theory developed in [7, 8, 9] shows that there exist stationary spatial solutions of the Gross-Pitaevskii (GP) equation which describe nonlinear waves supported by a supersonic BEC flow with Mach number

$$M = \frac{u}{c_s} > 1, \qquad (1)$$

where $u$ is velocity of the oncoming flow at $x \to -\infty$ and $c_s$ is the sound speed of the long-wavelength linear waves. The density $n$ of the condensate, as well as the components of the velocity field, depend on the variable

$$w = x - ay \qquad (2)$$

alone, where $a$ is the slope of the phase lines with respect to the $y$ axis and it is supposed that the velocity of the oncoming flow is directed along the $x$ axis. Then, the Mach cone for sound waves with infinitely large wavelength corresponds to the slope

$$a_M = \sqrt{M^2 - 1}, \qquad (3)$$

and it was shown in [9] that the spatial (oblique) solitons have $a > a_M$, that is they are located inside the Mach cone. In particular, it was shown that the shallow solitons are formed close to the Mach cone, $a - a_M \ll a_M$, and they are asymptotically described by the Korteweg-de Vries (KdV) equation; and deep solitons have $a \gg 1$ (i.e. they are formed at small angles with respect to direction of the oncoming flow) and are asymptotically described by the nonlinear Schrödinger (NLS) equation. On the contrary, the linear waves are generated outside the Mach cone and they have $a < a_M$. In fact, these linear waves had been observed in numerical simulations some years ago [10] but the theory of their generation has not been developed so far. The aim of this paper is to develop such a theory.

## LINEAR WAVES GENERATED IN A BEC FLOW PAST AN OBSTACLE

Our analysis is based on the use of the mean-filed description of BEC dynamics in the framework of the GP equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U(\mathbf{r})\psi + g|\psi|^2\psi, \qquad (4)$$

where $\psi(\mathbf{r})$ is the order parameter ("condensate wave function"), $U(\mathbf{r})$ is the potential which confines atoms of a Bose gas in a trap and/or describes interaction of the BEC with the obstacle, and $g$ is an effective coupling constant arising due to inter-atomic collisions with the $s$-wave scattering length $a_s$,

$$g = 4\pi\hbar^2 a_s/m, \qquad (5)$$

$m$ being the atomic mass. Here we consider the Bose-Einstein condensate with repulsive interaction between particles for which $g > 0$.

As suggested by the experimental set-up [6], we consider a two-dimensional flow of a condensate, so that the condensate wave function $\psi$ depends on only two spatial

coordinates, $\mathbf{r} = (x, y)$. To simplify the theory, we assume that the characteristic size of the obstacle is much less than its distance from the center of the trap, so that the oncoming flow can be considered as uniform with constant density $n_0$ of atoms and constant velocity $\mathbf{u}_0$ directed parallel to $x$ axis (see also estimates in [8]). It is convenient to transform Eq. (5) to a "hydrodynamic" form by means of the substitution

$$\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)} \exp\left(\frac{i}{\hbar} \int^{\mathbf{r}} \mathbf{u}(\mathbf{r}', t) d\mathbf{r}'\right), \qquad (6)$$

where $n(\mathbf{r}, t)$ is density of atoms in BEC and $\mathbf{u}(\mathbf{r}, t)$ denotes its velocity field, and to introduce dimensionless variables

$$\tilde{x} = x/\sqrt{2}\xi, \quad \tilde{y} = y/\sqrt{2}\xi, \quad \tilde{t} = (c_s/2\sqrt{2}\xi)t,$$
$$\tilde{n} = n/n_0, \quad \tilde{\mathbf{u}} = \mathbf{u}/c_s, \qquad (7)$$

where $\xi = \hbar/\sqrt{2mn_0 g}$ is the BEC healing length and numerical constants are introduced for future convenience. As a result of this transformation we obtain the system (we omit tildes for convenience of the notation)

$$\frac{1}{2}n_t + \nabla(n\mathbf{u}) = 0,$$
$$\frac{1}{2}\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} + \nabla n + \nabla\left[\frac{(\nabla n)^2}{8n^2} - \frac{\Delta n}{4n}\right] = 0, \qquad (8)$$

where $\nabla = (\partial_x, \partial_y)$. Since we shall consider waves far enough from the obstacle, the potential is omitted in (8).

We are interested in linear waves propagating on the background flow with $n = 1$, $u = M$, $v = 0$. Hence, we introduce

$$n = 1 + n_1, \quad u = M + u_1, \quad v = v_1, \qquad (9)$$

and linearize the system (8) with respect to small deviations $n_1, u_1, v_1$. As a result we obtain the system

$$\frac{1}{2}n_{1,t} + u_{1,x} + Mn_{1,x} + v_{1,y} = 0,$$
$$\frac{1}{2}u_{1,t} + Mu_{1,x} + n_{1,x} - \frac{1}{4}(n_{1,xxx} + n_{1,xyy}) = 0, \quad (10)$$
$$\frac{1}{2}v_{1,t} + Mv_{1,x} + n_{1,y} - \frac{1}{4}(n_{1,xxy} + n_{1,yyy}) = 0,$$

which describes propagation of linear waves in BEC with a uniform flow. We obtain the applicability condition of these equations, if we notice that in the linear wave $u_1 \sim Mn_1$ and the nonlinear terms of the order of magnitude $u_1\nabla u_1 \sim \nabla(Mu_1)^2$ can be neglected as long as they are much less than the linear ones $\sim \nabla n_1$. Thus, we get the criterion

$$n_1 \ll 1/M^2. \qquad (11)$$

Hence, if $M$ is large enough, the linear theory is applicable to description of waves outside the Mach cone far enough from the obstacle, where the density amplitude $n_1$ of the wave satisfies the condition (11). For harmonic waves $n_1, u_1, v_1 \propto \exp[i(k_x x + k_y y) - i\omega t]$ the system (10) yields at once the dispersion relation

$$\frac{\omega}{2} = Mk_x \pm k\sqrt{1 + \frac{k^2}{4}}, \qquad (12)$$

where $k = \sqrt{k_x^2 + k_y^2}$.

Now we consider the stationary wave patterns far enough from the obstacle where the condition (11) is supposed to be fulfilled. In fact, this problem is analogous to the Kelvin theory of ship waves generated by a ship moving in a deep water, but with a different dispersion law (12). Hence, we shall follow Kelvin's method in its form presented in [11, 12].

First, we notice that in a stationary wave pattern $\omega = 0$ and, hence, the components of the wavevector $\mathbf{k} = (k_x, k_y)$ are the functions of the space coordinates $(x, y)$ connected with each other by the relationship

$$G(k_x, k_y) \equiv Mk_x + k\sqrt{1 + \frac{k^2}{4}} = 0, \qquad (13)$$

where we have taken into account that for chosen geometry of the BEC flow the wave must propagate upwind i.e. $k_x < 0$.

Next, the "ship wave" pattern corresponds to a modulated two-dimensional wave where the wavevector $\mathbf{k}$ is a gradient of the phase [11],

$$\theta = \int_0^{\mathbf{r}} \mathbf{k} \cdot d\mathbf{r}. \qquad (14)$$

Hence, the components $(k_x, k_y)$ satisfy the condition

$$\frac{\partial k_x}{\partial y} - \frac{\partial k_y}{\partial x} = 0, \qquad (15)$$

which, with an account of (13), yields the equation for $k_y$

$$\frac{\partial k_y}{\partial x} - f'(k_y)\frac{\partial k_y}{\partial y} = 0, \qquad (16)$$

where $f'(k_y)$ is defined by the derivative of an implicit function (13):

$$f' = -\frac{\partial G/\partial k_y}{\partial G/\partial k_x}. \qquad (17)$$

It follows from Eq. (16) that $k_x$ and $k_y$ are constant along the characteristics defined as solutions of the equation

$$\frac{dy}{dx} = -f'(k_y). \qquad (18)$$

At last, since at large distances from the obstacle, the latter can be considered as a point-like source of waves,
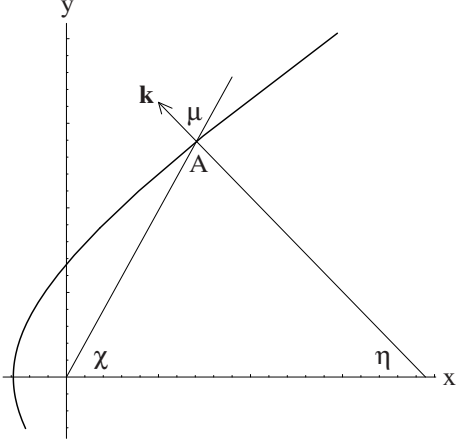
FIG. 1: Sketch of the wave front with wavevector $\mathbf{k}$ normal to it at the point $A$ and coordinates $\chi$ and $\eta$ defining the parameters of the wave pattern at this point.
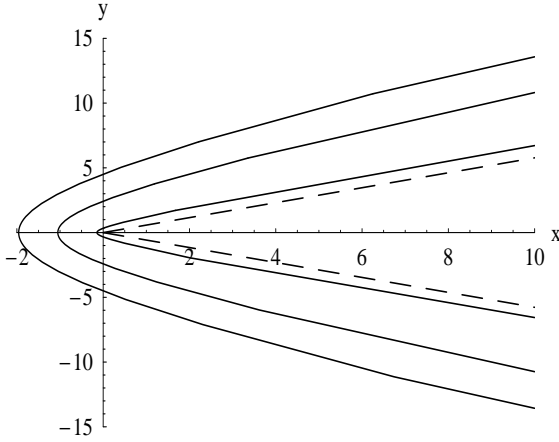


FIG. 2: Wave pattern of stationary linear waves generated in the flow of BEC past a point-like obstacle. Dashed line corresponds to the Mach cone of linear waves in the long wavelength limit.

the resulting stationary wave is centered, that is the characteristics intersect at the point $(x, y) = (0, 0)$ of the location of the obstacle. Hence, we obtain the solution

$$\frac{y}{x} = \tan\chi = -\frac{\partial G/\partial k_y}{\partial G/\partial k_x}, \qquad (19)$$

where $\chi$ is the angle between the $x$ axis and the radius-vector $\mathbf{r}$ of the point $A$ with wavevector $(k_x, k_y)$; see Fig. 1 where other convenient parameters are also defined, namely, the angle $\eta$ between the wavevector $\mathbf{k}$ and a negative direction of the $x$ axis, and the angle $\mu = \pi - \chi - \eta$ between vectors $\mathbf{k}$ and $\mathbf{r}$.

Thus, we have

$$k_x = -k\cos\eta, \qquad k_y = k\sin\eta, \qquad (20)$$

$$G(k, \eta) = -Mk\cos\eta + k\sqrt{1 + \frac{k^2}{4}} = 0. \qquad (21)$$

Then elementary calculation gives for (19) the expression

$$\tan\chi = \frac{(2M^2 - 1 - \tan^2\eta)\tan\eta}{(M^2 + 1)\tan^2\eta - (M^2 - 1)} \qquad (22)$$

and Eq. (21) yields

$$k = 2\sqrt{M^2\cos^2\eta - 1}. \qquad (23)$$

Thus, we have found that for a fixed value of $\eta$ the components $(k_x, k_y)$ are constant along the line $\chi = \text{const}$ with $\chi$ defined by Eq. (22) and the length of the wavevector given by (23). Therefore the phase (14) can be conveniently calculated by integration along the line $\chi = \text{const}$ with constant vector $\mathbf{k}$, so that

$$\theta = (k\cos\mu)r. \qquad (24)$$

It means that the lines of constant phase (e.g. the wave crests) $\theta$ are determined in parametrical form by Eq. (22) and

$$r = \frac{\theta}{k\cos\mu}, \qquad (25)$$

where $k$ is given by Eq. (23) and $\mu$ can be calculated from $\tan\mu = -\tan(\chi + \eta)$ which gives, after elementary algebra, the expression

$$\tan\mu = \frac{2M^2}{k^2}\sin 2\eta. \qquad (26)$$

This expression permits one to express Eq. (25) as

$$r = \frac{4\theta}{k^3}\sqrt{M^2(M^2 - 2)\cos^2\eta + 1} \qquad (27)$$

and Eq. (22) as

$$\tan\chi = \frac{(1 + k^2/2)\tan\eta}{M^2 - (1 + k^2/2)}. \qquad (28)$$

At last, the curves with constant phase $\theta$ are given in Cartesian coordinates by the formulas

$$\begin{aligned} x &= r\cos\chi = \frac{4\theta}{k^3}\cos\eta(1 - M^2\cos 2\eta), \\ y &= r\sin\chi = \frac{4\theta}{k^3}\sin\eta(2M^2\cos^2\eta - 1). \end{aligned} \qquad (29)$$

Thus, we have found the expressions describing the linear wave pattern in a parametric form where the parameter $\eta$ changes in the interval

$$-\arccos\frac{1}{M} \le \eta \le \arccos\frac{1}{M}. \qquad (30)$$

For $\eta = 0$ we have $x < 0$ and $y = 0$, that is small values of $\eta$ correspond to the wave before the obstacle. The

boundary values $\eta = \pm \arccos(1/M)$ correspond to the lines

$$\frac{x}{y} = \pm\sqrt{M^2 - 1} = \pm a_M, \qquad (31)$$

that is the curves of constant phase become asymptotically the straight lines parallel to the Mach cone. The general pattern is shown in Fig. 2.

## NUMERICAL SIMULATIONS

We have compared the above approximate analytical theory with the exact numerical solution of the GP equation, which in non-dimensional units (7) takes the standard form

$$i\psi_t = -\tfrac{1}{2}(\psi_{xx} + \psi_{yy}) + U(x,y)\psi + |\psi|^2\psi, \qquad (32)$$

which corresponds for $U = 0$ to the system (8) with

$$\psi = \sqrt{n}\exp\left(i\int^{\mathbf{r}}\mathbf{u}(\mathbf{r}',t)d\mathbf{r}'\right).$$

In our simulations the obstacle was modeled by an impenetrable disk with radius $r = 1$. Such an obstacle introduces large enough perturbation into the flow to generate an oblique solitons pair behind it (see [8]). We assume that at the initial moment $t = 0$ there is no disturbance in the condensate, so that it is described by the plane wave function

$$\psi(x,y)|_{t=0} = \exp(iMx)$$

corresponding to a uniform condensate flow. For large enough evolution time, the wave pattern around the obstacle tends to a stationary structure. An example of such a structure for $M = 2$ is shown in Fig. 2. A dashed line corresponds to the analytic theory developed in the preceding Section; as we see it agrees very well with the numerical results for $M = 2$.

The condition (11) indicates that the nonlinear effects grow up with increase of $M$. To demonstrate this explicitly, we have compared the wavelength $\lambda$ at $y = 0$ calculated using the developed linear analytic theory with the same parameter obtained from our full numerical simulations. According to linear theory, the wavelength at $y = 0$ (i.e. $\eta = 0$) is constant and equal to

$$\lambda = \frac{2\pi}{k} = \frac{\pi}{\sqrt{M^2 - 1}}, \qquad (33)$$

where we have used Eq. (23). In Fig. 4 we compare this dependence of the wavelength $\lambda$ on the Mach number $M$ with the results of numerical simulations at the point with $n_1 \approx 0.1$. As we see, Eq. (33) is very accurate for values of $M$ satisfying the condition (11) and discrepancy between analytical and numerical results slightly increases with increase of $M$. In general, this plot confirms validity of a linear theory in the region of its applicability.
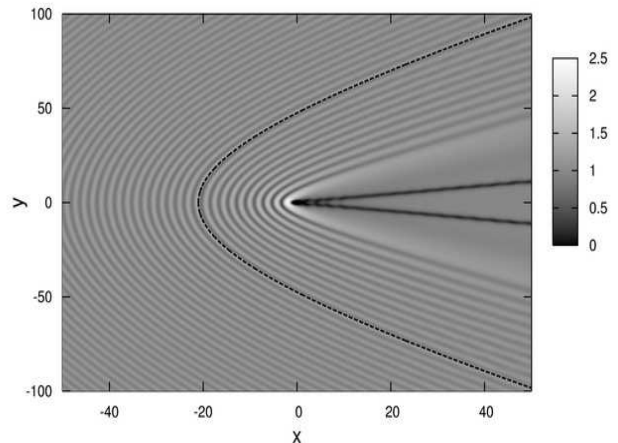


FIG. 3: Numerically calculated wave pattern of stationary linear waves generated in the flow of BEC past an obstacle. The Mach number is equal to $M = 2$ and the radius of an impenetrable obstacle to $r = 1$. Dashed line corresponds to the linear analytical theory for the line of constant phase. It corresponds to Eqs. (29) but the curve is shifted to 1 unit of length to the left from the center of the obstacle for better fitting to numerics. It should be noted that this distance is negligibly small in our theory corresponding to a point-like obstacle. A pair of oblique dark lines behind the obstacle correspond to spatial solitons studied in [8].
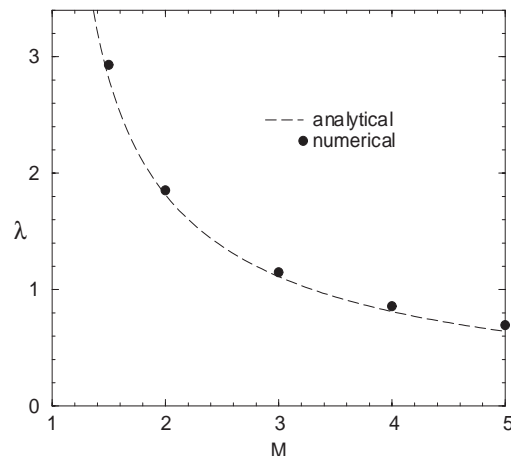


FIG. 4: Dependence of the wavelength of the wave generated before the obstacle on the Mach number calculated numerically (full time-dependent GP equation)—dots, and analytically (linear theory)—dashed line.

## CONCLUSION

We have developed here the theory of linear waves generated by the flow of BEC past an obstacle. The linear approximation is correct for small enough amplitudes of the perturbation. This condition is satisfied in the case of small disturbance introduced by the obstacle and not too high values of the Mach number. Our numerical simulations confirm the analytical theory in the region of its

applicability.

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